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The Derivatives and Integrals of Fractional Order in Walsh–Fourier Analysis, with Applications to Approximation Theory

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In recent years the derivatives and integrals introduced by P. L. Butzer and H. J. Wagner [1] have been widely discussed in the literature. In Walsh–Fourier analysis and approximation theory they play a very important role. In 1977, C. W. Onneweer [4] introduced a kind of derivative of fractional order and obtained some intersting results, but they are not a generalization of the preceding derivatives. The aim of the present paper is to generalize the Butzer and Wagner derivatives and integrals to the case of fractional order. In this paper we give their definition and, under this new definition, establish some of the basic properties given in the case of integral order by Butzer and Wagner [1], Pal and Simon [6], etc. As applications to approximation theory, we generalize a Jackson-type theorem and Bernstein-type theorem given by Watari [10], Butzer and Wagner [2, 8] to the case of fractional order, and estimate the degree of approximation by the typical means of the Walsh–Fourier series.

The generalized "Walsh functions" studied in this paper were also considered by Виленкин [13].

1. SYMBOLS

(a) Let $\mathbb{N} := \{1, 2,...\}$; $\mathbb{P} := \{0, 1, 2,...\}$; $m_j \in \mathbb{N} - \{1\}$, $j \in \mathbb{P}$, $\overline{\lim}_{j\to\infty} m_k < \infty$; $M_0 := 1$, $M_r := m_0 m_1 \cdots m_{r-1}$, $r \in N$; $\mathbb{Z}_j := \{0, 1,..., m_j - 1\}$; then each $x \in [0, 1)$ has a unique expansion $x = \sum_{j=1}^{\infty} x_j M_j^{-1}$ $(x_j \in \mathbb{Z}_j)$, and each $k \in \mathbb{P}$ has a unique expansion $k = \sum_{j=0}^{\infty} k_j M_j$ $(k \in \mathbb{Z}_j)$.

(b) Let $X = \sum_{j=1}^{\infty} x_j M_j^{-1}$, $y = \sum_{j=1}^{\infty} y_j M_j^{-1}$ $(x_j, y_j \in \mathbb{Z}_j)$.

$$x \oplus y := \sum_{j=1}^{\infty} (x_j + y_j - \alpha_j) M_j^{-1},$$

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where

$$\alpha_j = 0 \quad \text{if} \quad x_j + y_j < m_j,$$
$$= m_j \quad \text{if} \quad x_j + y_j \geqslant m_j.$$

If $m_j = m$, $j \in \mathbb{P}$, then the symbol $\oplus$ is called the addition modulo m.

(c) $\varphi_k(x) := \exp(2\pi i)/(m_k)\,\chi_{k+1}$ $\quad (x \in [0, 1), \quad i = \sqrt{-1}, \quad k \in \mathbb{P})$; $\psi_k(x) := \prod_{j=0}^{\infty} (\varphi_j(x))^{k_j}$, where $k = \sum_{j=0}^{\infty} k_j M_j$, $k_j \in Z_j$. $D_n(x) := \sum_{j=0}^{n-1} \bar{\psi}_j(x)$, $F_n(x) := 1/n \sum_{j=1}^{n} D_j(x)$.

(d) $X := X[0, 1) := WC[0, 1)$ or $L^p[0, 1)$ $(1 \leqslant p < \infty)$,

$$\|f\|_X := \sup_{0 \leqslant x < 1} |f(x)|, \qquad \text{if} \quad X = WC,$$
$$:= \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}, \qquad \text{if} \quad X = L^p,$$

where $WC[0, 1) := \{f \mid \sup_{0 \leqslant x < 1} |f(x \oplus h) - f(x)| \to 0 \text{ as } h \to 0\}$.

(e) If $f \in X[0, 1)$, $g \in L^1[0, 1)$, $(f * g)(x) := \int_0^1 f(x \oplus t) f(t)\, dt$.

(f) For $f \in X[0, 1)$, let $f(k) := \int_0^1 f(t)\, \overline{\psi_k(t)}\, dt$.

(g) $\omega(\delta) := \omega(f, \delta) := \omega(X, f, \delta) := \sup_{0 \leqslant h < \delta} \|f(\cdot \oplus h) - f(\cdot)\|_X$. $\operatorname{Lip} \beta := \operatorname{Lip}(X, \beta) := \{f \in X \mid \omega(X, f, \delta) = O(\delta^\beta),\ \delta \to 0\}$. $\operatorname{lip} \beta := \operatorname{lip}(X, \beta) := \{f \in X \mid \omega(X, f, \delta) = o(\delta^\beta),\ \delta \to 0)$. $E_n(f) := E_n(X, f) := \inf_{P_n \in W_n} \|f - P_n\|_X$, where W_n denotes the set of all Walsh polynomials of order $\leqslant n$, i.e., $W_n := \{f \in L^1[0, 1) \mid f^\wedge(k) = 0,\ k \geqslant n\}$.

2. Definition and Properties of Derivative (Integral)

In [1], Butzer and Wagner gave the following definition of the dyadic derivative:

Definition A. Let $f \in X[0, 1)$,

$$d_n f(x) = \frac{1}{2} \sum_{j=0}^{n} 2^j \left[f(x) - f\left(x \oplus \frac{1}{2^{j+1}}\right) \right], \tag{1}$$

where $\oplus$ is the addition modulo 2; if there exists $g \in X[0, 1)$ such that $\lim_{n \to \infty} \|d_n f(\cdot) - f(\cdot)\|_X = 0$, then g is called the (strong) derivative of f.

In [7, 11] Zhen Weixing *et al.* gave the following definition of the m-adic derivative:

DEFINITION B. Let $f \in X[0, 1)$,

$$d_n f(x) = \sum_{k=0}^{n} m^k \sum_{j=0}^{m-1} a_j f(x \oplus jm^{-k-1}), \tag{2}$$

where $\oplus$ is the addition modulo m, and

$$\begin{aligned} a_j &= \frac{m-1}{2}, && j = 0, \\ &= \frac{1}{\exp \frac{-2\pi i}{m} j - 1}, && j = 1, 2, \dots, m-1. \end{aligned}$$

If there exists $g \in X[0, 1)$ such that $\lim_{n \to \infty} \| d_n f(\cdot) - g(\cdot) \| = 0$, then g is called the (strong) derivative of f.

In [5] Onneweer gave the following definition of the $\{m_j\}$-adic derivative:

DEFINITION C. Let $f \in X[0, 1)$,

$$d_n f(x) = \sum_{j=0}^{n} M_j \sum_{l=0}^{m_j-1} l m_j^{-1} \sum_{k=0}^{m_j-1} \overline{\varphi_j \left(\frac{k}{M_{j+1}} \right)}^{\,l} f\left(x \oplus \frac{k}{M_{j+1}} \right); \tag{3}$$

if there exists a g such that $\lim_{n \to \infty} \| d_n f(\cdot) - g(\cdot) \| = 0$, then g is called the (strong) derivative of f, and denoted by $D^{[1]} f$.

Formula (3) may be reduced to

$$d_n f(x) = \sum_{j=0}^{n-1} M_j \sum_{k=0}^{m_j-1} a_{jk} f\left(x \oplus \frac{k}{M_{j+1}} \right), \tag{4}$$

where

$$\begin{aligned} a_{jk} &= \frac{m_j - 1}{2}, && k = 0, j = 0, 1, 2, \dots, n-1, \\ &= \frac{1}{\exp \frac{-2\pi i}{m_j} k - 1}, && k = 1, 2, \dots, m_j - 1, j = 0, 1, \dots, n-1. \end{aligned}$$

In fact,

$$\sum_{l=0}^{m_j-1} l/m_j \overline{\varphi_j \left(\frac{k}{M_{j+1}} \right)}^{\,l} = 1/m_j \sum_{l=0}^{m_j-1} l e^{-(2\pi i/m_j)kl} = a_{jk},$$

$$j = 0, 1, \dots, n-1, \quad k = 0, 1, \dots, m_j - 1.$$

Obviously, Definition B is more general than Definition A, and Definition C than Definition B.

In [1] Butzer and Wagner also gave a definition of the integral for the case $m_j = 2$, $j \in \mathbb{P}$; it is also suited to that of m_j in the general case. The integral of f can be defined as $I^{[1]}f := W_1 * f$, where $W_1 \in L^1(0, 1)$ and

$$\begin{aligned} \hat{W_1}(k) &= 1 && \text{if} \quad k = 0, \\ &= k^{-1} && \text{if} \quad k \in N. \end{aligned}$$

Now we give a definition of derivative and integral of fractional order, which is a generalization of the preceding definitions of derivative and integral of integral order.

To simplify our statements hereafter, in this paper we define $0^\alpha = 1$ if $\alpha \leqslant 0$.

DEFINITION 1. Let $f \in X(0, 1)$, $\alpha \in \mathbb{R}$, $T_r^{\langle\alpha\rangle}(t) := \sum_{k=0}^{M_r - 1} k^\alpha \overline{\psi_k(t)}$; if there exists $g \in X[0, 1)$ such that $\lim_{r \to \infty} \|(T_r^{\langle\alpha\rangle} * f)(\cdot) - g(\cdot)\|_X = 0$, then if $\alpha > 0$, g is called the (strong) derivative of order α of f in $X[0, 1)$; if $\alpha < 0$, g is called the (strong) integral of order $(-\alpha)$ of f in $X[0, 1)$. In both cases, g will be denoted by $T^{\langle\alpha\rangle}f$.

THEOREM 1.

(1) $T^{\langle\alpha\rangle}$ *are linear operators for* $\alpha \in \mathbb{R}$.

(2) $T^{\langle\alpha\rangle}\psi_k = k^\alpha \psi_k, \qquad k \in \mathbb{P}$.

(3) *If* $T^{\langle\alpha\rangle}f \in X[0, 1)$, *then* $(T^{\langle\alpha\rangle}f)\hat{}\,(k) = k^\alpha f\hat{}\,(k)$, $k \in \mathbb{P}$.

(4) *If* $\alpha > 0$, $T^{\langle\alpha\rangle}f = 0 \Leftrightarrow f = \text{const.}$; *if* $\alpha = 0$, $T^{\langle\alpha\rangle}f = 0 \Leftrightarrow f = 0$.

Proof. (1) and (2) are trivial in view of the definition.

(3) If $T^{\langle\alpha\rangle}f \in X[0, 1)$, i.e., $\|(f * T_r^{\langle\alpha\rangle})(\cdot) - T^{\langle\alpha\rangle}f(\cdot)\|_X \to 0$ $(r \to \infty)$, then $(T^{\langle\alpha\rangle}f)\hat{}\,(k) = k^\alpha f\hat{}\,(k)$, $k \in \mathbb{P}$.

(4) Assume $\alpha > 0$; if $T^{\langle\alpha\rangle}f = 0$, then $k^\alpha f\hat{}\,(k) = 0$, $k \in \mathbb{P}$, i.e., $f\hat{}\,(k) = 0$, $k \in \mathbb{N}$, thus $f = $ const. Conversely, if $f = $ const, then by definition, $T^{\langle\alpha\rangle}f = 0$. For $\alpha < 0$, the proof is analogous.

LEMMA 1. *If* $\alpha < 0$, *then* $T_r^{\langle\alpha\rangle}(t)$ *converges to a function* $T_\infty^{\langle\alpha\rangle}(t)$ *in* $X[0, 1)$, *and*

$$(T_\infty^{\langle\alpha\rangle})\hat{}\,(k) = k^\alpha, \quad k \in \mathbb{P}; \qquad \| T_r^{\langle\alpha\rangle}(\cdot) - T_\infty^{\langle\alpha\rangle}(\cdot)\|_{L^1} = O(M_r^\alpha) \ (r \to \infty).$$

Proof. Applying Abel's transform twice, we get, assuming $s > r \geqslant 0$,

$$
\begin{aligned}
&T_s^{\langle\alpha\rangle}(t) - T_r^{\langle\alpha\rangle}(t)\\
&\quad = \sum_{k=M_r}^{M_s-1} k^\alpha \psi_k(t) = \sum_{k=M_r}^{M_s-3} [k^\alpha - 2(k+1)^\alpha + (k+2)^\alpha](k+1) F_{k+1}(t)\\
&\quad - [M_r^\alpha - (M_r+1)^\alpha] M_r F_{M_r}(t)\\
&\quad + [(M_s-2)^\alpha - (M_s-1)^\alpha](M_s-1) F_{M_s-1}(t)\\
&\quad - M_r^\alpha D_{M_r}(t) + (M_s-1)^\alpha D_{M_s}(t).
\end{aligned}
$$

On the other hand (see [6]), since

$$\|D_{M_r}(\cdot)\|_{L^1} = 1; \qquad \|F_k(\cdot)\|_{L^1} = O(1);$$

$$M_r^\alpha - (M_r+1)^\alpha = O(M_r^{\alpha-1}); \qquad (M_s-2)^\alpha - (M_s-1)^\alpha = O(M_s^{\alpha-1});$$

$$\sum_{k=M_r}^{M_s-3} [k^\alpha - 2(k+1)^\alpha + (k+1)^\alpha](k+1) = O(1) \sum_{k=M_r}^{M_s-3} k^{\alpha-1} = O(M_r^\alpha).$$

Thus,

$$\|T_s^{\langle\alpha\rangle}(\cdot) - T_r^{\langle\alpha\rangle}(\cdot)\|_{L^1} = O(M_r^\infty) \to 0 \qquad \text{as} \quad r, s \to \infty. \tag{5}$$

Therefore by the completeness of $L'[0, 1)$, there exists $T_\infty^{\langle\alpha\rangle}(t) \in L'[0, 1)$ such that

$$\|T_r^{\langle\alpha\rangle}(\cdot) - T_\infty^{\langle\alpha\rangle}(\cdot)\|_{L^1} \to 0 \qquad \text{as} \quad r \to \infty. \tag{6}$$

It is obvious by (5) and (6) that $(T_\infty^{\langle\alpha\rangle})\hat{}(k) = k^\alpha$ and $\|T_r^{\langle\alpha\rangle}(\cdot) - T_\infty^{\langle\alpha\rangle}(\cdot)\|_{L^1} = O(M_r^\alpha)$.

The following simple lemma is a counterpart of Lemma 2 in [2]; it plays an important role in this paper.

LEMMA 2. *Assume $f \in X(0, 1)$ and let $W(t)$ be a Walsh polynomial of order $\leqslant M_r$; then*

$$\left\| \int_0^1 f(\cdot \oplus t)\, \overline{\varphi_r(t)}^k W(t)\, dt \right\|_X = \omega\left(X, f, \frac{1}{M_r}\right) \|W(t)\|_{L^1}, \quad k \in \{1, 2, ..., m_r - 1\}.$$

Proof. Let $x \in [0, 1)$, $j \in \mathbb{Z}_r$, then

$$
\begin{aligned}
I &= \int_0^1 f(x \oplus t)\, \overline{\varphi_r(t)}^k W(t)\, dt\\
&= \int_0^1 f\left(x \oplus t \oplus \frac{j}{M_{r+1}}\right) \overline{\varphi_r\left(t \oplus \frac{j}{M_{r+1}}\right)}^k W\left(t \oplus \frac{j}{M_{r+1}}\right) dt\\
&= \int_0^1 f\left(x \oplus t \oplus \frac{j}{M_{r+1}}\right) \overline{\varphi_r(t)}^k e^{-(2\pi i/m_r)jk} W(t)\, dt,
\end{aligned}
$$

i.e.,

$$I=\int_0^1 \frac{1}{m_r}\sum_{j=0}^{m_r-1} e^{-(2\pi i/m_r)jk}\,\overline{\varphi_r(t)}^k f\left(x\oplus t\oplus\frac{j}{M_{n+1}}\right) W(t)\,dt.$$

Thus,

$$\begin{aligned}\|I\|_X &\leqslant \frac{1}{m_r}\left\|\sum_{j=0}^{m_r-1} e^{-(2\pi i/m_r)jk} f\left(\cdot\oplus\frac{j}{M_{r+1}}\right)\right\|_X \|W(t)\|_{L^1}\\ &=\frac{1}{m_r}\left\|\sum_{j=0}^{m_r-1} e^{-(2\pi i/m_r)jk}\left[f\left(\cdot\oplus\frac{j}{M_{r+1}}\right)-f(\cdot)\right]\right\|_X\cdot\|W(t)\|_{L^1}\\ &\leqslant \omega\left(X,f,\frac{1}{M_r}\right)\|W\|_{L^1}.\end{aligned}$$

LEMMA 3. *Assume* $f\in X[0,1)$, $\alpha\geqslant 0$, $s>r\geqslant 0$; *then* $\|T_s^{\langle\alpha\rangle}*f-T_r^{\langle\alpha\rangle}*f\|_X=O(1)\sum_{l=r}^{s-1}\omega(f,1/M_l)\,M_l^\alpha$. *In particular,* $\|T_s^{\langle\alpha\rangle}*f\|_X=O(1)\sum_{l=0}^{s-1}\omega(f,1/M_l)\,M_l^\alpha$.

Proof.

$$\begin{aligned}&\|T_s^{\langle\alpha\rangle}*f-T_r^{\langle\alpha\rangle}*f\|_X\\ &\quad=\left\|\int_0^1 f(\cdot\oplus t)\sum_{l=r}^{s-1}\sum_{j=1}^{m_l-1}\sum_{k=jM_l}^{(j+1)M_l-1} k^\alpha\,\overline{\psi_k(t)}\,dt\right\|_X\\ &\quad=\left\|\int_0^1 f(\cdot\oplus t)\sum_{l=r}^{s-1}\sum_{j=1}^{m_l-1}\sum_{k=0}^{M_l-1}(jM_l+k)^\alpha\,\overline{\psi_{jM_l+k}(t)}\,dt\right\|_X\\ &\quad=\left\|\sum_{l=r}^{s-1}\sum_{j=1}^{m_l-1}\int_0^1 f(\cdot\oplus t)\,\overline{\varphi_l(t)}^j\sum_{k=0}^{M_l-1}(jM_l+k)^\alpha\,\overline{\psi_k(t)}\,dt\right\|_X\\ &\quad\leqslant\sum_{l=r}^{s-1}\sum_{j=1}^{m_l-1}\omega\left(f,\frac{1}{M_l}\right)\left\|\sum_{k=0}^{M_l-1}(jM_l+k)^\alpha\,\overline{\psi_k(t)}\right\|_{L^1}.\end{aligned}$$

By applying Abel's transform twice, we have

$$\begin{aligned}&\left\|\sum_{k=0}^{M_l-1}(jM_l+k)^\alpha\,\overline{\psi_k(t)}\right\|_{L^1}\\ &\quad=\left\|\sum_{k=0}^{M_l-3}[(jM_l+k)^\alpha-2(jM_l+k+1)^\alpha+(jM_l+k+2)^\alpha](k+1)\,F_{k+1}(t)\right.\\ &\qquad+[((j+1)\,M_l-2)^\alpha-[(j+1)\,M_l-1)^\alpha](M_l-1)\,F_{M_l-1}(t)\\ &\qquad\left.-[(j+1)\,M_l-1]^\alpha D_{M_l}(t)\right\|_{L^1}=O(M_l^\alpha).\end{aligned}\tag{7}$$

Therefore, $\| T_s^{\langle\alpha\rangle} * f - T_r^{\langle\alpha\rangle} * f\|_X = O(1) \sum_{l=r}^{s-1} \omega(f, 1/M_l)\, M_l^\alpha$.

LEMMA 4. *If* $f \in X[0,1]$, $\alpha > 0$, $M_s \leqslant n < M_{s+1}$, $g(t) = \sum_{k=a_sM_s}^{n-1} (n^\alpha - k^\alpha)\,\overline{\psi_k(t)}$, *then* $\|(g * f)(\cdot)\|_X = O(1)\,\omega(X, f, 1/M_s)\, M_s^\alpha$.

Proof. Let $n = \sum_{j=0}^{s} a_j M_j$ ($a_j \in \mathbb{Z}_j$, $j = 0, 1,..., s$, $a_s \neq 0$). Then

$$\begin{aligned}
g(t) &= \sum_{k=a_sM_s}^{n-1} (n^\alpha - k^\alpha)\,\overline{\psi_k(t)} \\
&= \sum_{r=1}^{s} \sum_{j=0}^{a_{s-r}-1} \sum_{k=a_sM_s+a_{s-1}M_{s-1}+\cdots+a_{s-r+1}M_{s-r+1}+jM_{s-r}}^{a_sM_s+a_{s-1}M_{s-1}+\cdots+a_{s-r+1}M_{s-r+1}+(j+1)M_{s-r}-1} (n^\alpha - k^\alpha)\,\overline{\psi_k(t)} \\
&= \sum_{r=1}^{s} \overline{\varphi_s(t)}^{a_s} \cdots \overline{\varphi_{s-r+1}(t)}^{a_{s-r+1}} \sum_{j=0}^{a_{s-r}-1} \overline{\varphi_{s-r}(t)}^{j} \\
&\quad\times \sum_{k=0}^{M_{s-r}-1} [n^\alpha - (a_sM_s + \cdots + a_{s-r+1}M_{s-r+1} + jM_{s-r} + k)^\alpha]\,\overline{\psi_k(t)}.
\end{aligned}$$

By Lemma 2, we have

$$\begin{aligned}
&\|(g * f)(t)\|_X \\
&\leqslant \omega\left(X, f, \frac{1}{M_s}\right) \sum_{r=1}^{s} \sum_{j=0}^{a_{s-r}-1} \\
&\quad\times \left\| \sum_{k=0}^{M_{s-r}-1} [n^\alpha - (a_sM_s + \cdots + a_{s-r+1}M_{s-r+1} + jM_{s-r} + k)^\alpha]\,\overline{\psi_k(t)} \right\|_{L^1} \\
&= \omega\left(X, f, \frac{1}{M_s}\right) \sum_{r=1}^{s} \sum_{j=0}^{a_{s-r}-1} \\
&\quad\times \left\| \sum_{k=0}^{M_{s-r}-1} [2(a_sM_s + \cdots + a_{s-r+1}M_{s-r+1} + jM_{s-r} + k - 1)^\alpha \right. \\
&\quad - (a_sM_s + \cdots + a_{s-r+1}M_{s-r+1} + jM_{s-r} + k)^\alpha \\
&\quad - (a_sM_s + \cdots + a_{s-r+1}M_{s-r+1} + k + 2)^\alpha](k+1)\,F_{k+1}(t) \\
&\quad + [(a_sM_s + \cdots + a_{s-r+1}M_{s-r+1} + (j+1)\,M_{s-r} - 1)^\alpha \\
&\quad - (a_sM_s + \cdots + a_{s-r+1}M_{s-r+1} + (j+1)\,M_{s-r} - 2)^\alpha](M_{s-r} - 1)\,F_{M_{s-r}}(t) \\
&\quad \left. + [n^\alpha - (a_sM_s + \cdots + a_{s-r+1}M_{s-r+1} + (j+1)\,M_{s-r} - 1)^\alpha]\,D_{M_{s-r}}(t) \right\|_{L^1} \\
&= O(1)\,\omega\left(X, f, \frac{1}{M_s}\right) \sum_{r=1}^{s} (M_s^{\alpha-2}M_{s-r}^2 + M_s^{\alpha-1}M_{s-r} + M_s^{\alpha+1}M_{s-r}) \\
&= O(1)\,\omega\left(X, f, \frac{1}{M_s}\right) M_s^\alpha.
\end{aligned}$$

LEMMA 5. *If* $T^{\langle\alpha\rangle}f \in X[0,1)$, $\alpha > 0$, *then*

(1) $\omega(f, 1/M_r) = O(1)\, 1/M_r^\alpha \omega(T^{\langle\alpha\rangle}f, 1/M_r)$.

(2) $\omega(T^{\langle\alpha\rangle}f, 1/M_r) = O(1) \sum_{l=r}^{\infty} \omega(f, 1/M_r) M_l^\alpha$.

Proof. (1) Let $0 \leqslant h < 1/M_r$. By Theorem 1 (3), we have

$$|f(x \oplus h) - f(x)|\hat{}(k) = |(T_\infty^{\langle -\alpha\rangle} - T_r^{\langle -\alpha\rangle}) * (T^{\langle\alpha\rangle}f(x \oplus h) - T^{\langle\alpha\rangle}f(x))|\hat{}(k),$$
$$k \in \mathbb{P},$$

i.e.,

$$f(x \oplus h) f(x) = (T_\infty^{\langle -\alpha\rangle} - T_r^{\langle -\alpha\rangle}) * (T^{\langle\alpha\rangle}f(x \oplus h) - T^{\langle\alpha\rangle}f(x)).$$

Therefore, by Lemma 1, we have $\omega(f, 1/M_r) = O(1)(1/M_r^\alpha)\, \omega(T^{\langle\alpha\rangle}f, 1/M_r)$.

(2) Since $\omega(T^{\langle\alpha\rangle}f, 1/M_r) \leqslant 2E_{M_r}(T^{\langle\alpha\rangle}f) \leqslant 2\,\| T^{\langle\alpha\rangle}f - T_r^{\langle\alpha\rangle} * f\|_X \leqslant 2\,\|T^{\langle\alpha\rangle}f - T_s^{\langle\alpha\rangle} * f\|_X + 2\,\|T_s^{\langle\alpha\rangle} * f - T_s^{\langle\alpha\rangle} * f - T_r^{\langle\alpha\rangle} * f\|_X$, by Lemma 3, we get for $s \to \infty$. $\omega(T^{\langle\alpha\rangle}f, 1/M_r) = O(1) \sum_{l=r}^{\infty} \omega(f, 1/M_l) M_l^\alpha$.

LEMMA 6. *If for* $f \in X[0,1)$ *there exists* $g \in X[0,1)$ *such that* $k^\alpha f\hat{}(k) = g\hat{}(k)$, $\alpha > 0$, $k \in \mathbb{P}$, *then* $f = T_\infty^{\langle -\alpha\rangle} * g + f\hat{}(0)$.

Proof. Since $(T_\infty^{\langle -\alpha\rangle} * g)\hat{}(k) = k^{-\alpha} k^\alpha f\hat{}(k)$, we have $f = T_\infty^{\langle -\alpha\rangle} * g + f\hat{}(0)$.

THEOREM 2. *If* $\alpha < 0$, $f \in X[0,1)$, *then* $T^{\langle\alpha\rangle}f \in X[0,1)$ *and* $T^{\langle\alpha\rangle}f = T^{\langle\alpha\rangle}f = T_\infty^{\langle\alpha\rangle} * f$.

Proof. By Lemma 1, $T_\infty^{\langle\alpha\rangle} \in L^1[0,1)$, thus $T_\infty^{\langle\alpha\rangle} * f \in X[0,1)$. Moreover,

$$\|T_r^{\langle\alpha\rangle} * f - T_\infty^{\langle\alpha\rangle} * f\|_X \leqslant \|T_r^{\langle\alpha\rangle} - T_\infty^{\langle\alpha\rangle}\|_{L^1} \|f\|_X = O(M_r^\alpha) \to 0 \qquad (r \to \infty).$$

This completes the proof.

By this theorem and Lemma 1 we get immediately

COROLLARY 2.1. $T^{\langle -1\rangle}f = I^{[1]}f$, *where* $I^{[1]}$ *is the integral operator introduced by Butzer and Wagner.*

THEOREM 3. *If* $\alpha > 0$, $U^{\langle\alpha\rangle} := \{f \in X[0,1) \mid T^{\langle\alpha\rangle}f \in X[0,1)\}$;

$$U_1^{\langle\alpha\rangle} := \left\{f \in X[0,1) \mid \omega\left(f, \frac{1}{M_l}\right) = O\left(\frac{a_l}{M_l^\alpha}\right)\right\}, \quad \textit{where } a_l > 0, \sum_{l=1}^{\infty} a_l < \infty;$$

$$U_2^{\langle\alpha\rangle} := \left\{f \in X[0,1) \mid \omega\left(f, \frac{1}{M_l}\right) = O(M_l^{-\alpha})\right\},$$

then $U_1^{\langle\alpha\rangle} \subset U^{\langle\alpha\rangle} \subset U_2^{\langle\alpha\rangle}$.

Proof. Suppose $f \in U_1^{\langle\alpha\rangle}$. By Lemma 3, we have $\|T_s^{\langle\alpha\rangle} * f - T_r^{\langle\alpha\rangle} * f\|_X = O(1) \sum_{l=r}^{s-1} \omega(f, 1/M_l) M_l^\alpha = O(1) \sum_{l=r}^{s+1} a_l \to 0 \quad (r, s \to \infty)$. By the completeness of $X[0, 1)$, there exists $g \in X[0, 1)$ such that $\|T_r^{\langle\alpha\rangle} * f - g\|_X \to 0$ $(r \to \infty)$, i.e., $f \in U^{\langle\alpha\rangle}$.

Suppose $f \in U^{\langle\alpha\rangle}$. By Lemma 5, we have $\omega(f, 1/M_r) = O(1)(1/M_r^\alpha)$ $\omega(T^{\langle\alpha\rangle}f, 1/M_r) = O(1/M_r^\alpha)$, i.e., $f \in U_2^{\langle\alpha\rangle}$.

COROLLARY 3.1. *If $0 < \alpha < \beta$, then $U^{\langle\beta\rangle} \subset \operatorname{lip} \beta \subset U^{\langle\alpha\rangle} \subset \operatorname{lip} \alpha$.*

THEOREM 4. *Let $f \in X[0, 1)$, $\int_0^1 f(x)\,dx = 0$. If one of the following two conditions holds*, then $T^{\langle\alpha\rangle}T^{\langle\beta\rangle}f = T^{\langle\alpha+\beta\rangle}f$.

(1) $\alpha \leqslant 0$ *and* $T^{\langle\beta\rangle}f \in X[0, 1)$.

(2) $\alpha > 0$ *and* $T^{\langle\alpha+\beta\rangle}f \in X[0, 1)$ *or* $T^{\langle\beta\rangle}f$, $T^{\langle\alpha\rangle}T^{\langle\beta\rangle}f \in X[0, 1)$

(*cf. Theorem* 3 *and Corollary* 4 *in* [4].)

Proof. (1) If $\alpha \leqslant 0$, $T^{\langle\beta\rangle}f \in X[0, 1)$, then by Theorem 2 and Corollary 3.1, we have $T^{\langle\alpha\rangle}T^{\langle\beta\rangle}f$, $T^{\langle\alpha+\beta\rangle}f \in X[0, 1)$. Therefore, since $(T^{\langle\alpha\rangle}T^{\langle\beta\rangle}f)\hat{}(k) = k^{\alpha+\beta}f\hat{}(k) = (T^{\langle\alpha+\beta\rangle}f)\hat{}(k)$, $k \in \mathbb{P}$, we have $T^{\langle\alpha\rangle}T^{\langle\beta\rangle}f = T^{\langle\alpha+\beta\rangle}f$.

(2) If $\alpha > 0$, $T^{\langle\alpha+\beta\rangle}f \in X[0, 1)$, then $T^{\langle\beta\rangle}f \in X[0, 1)$, thus by $T_r^{\langle\alpha\rangle} * T^{\langle\beta\rangle}f = T_r^{\langle\alpha+\beta\rangle}f$ and $T^{\langle\alpha+\beta\rangle}f \in X[0, 1)$, we get $T^{\langle\alpha\rangle}T^{\langle\beta\rangle}f \in X[0, 1)$. If $T^{\langle\beta\rangle}f$, $T^{\langle\alpha\rangle}T^{\langle\beta\rangle}f \in X[0, 1)$, then by $T_r^{\langle\alpha\rangle} * T^{\langle\beta\rangle}f = T_r^{\langle\alpha+\beta\rangle}f$, we know $T^{\langle\alpha+\beta\rangle}f \in X[0, 1)$. Therefore, since $(T^{\langle\alpha\rangle}T^{\langle\beta\rangle}f)\hat{}(k) = k^{\alpha+\beta}f\hat{}(k) = (T^{\langle\alpha+\beta\rangle}f)\hat{}(k)$, we get $T^{\langle\alpha\rangle}T^{\langle\beta\rangle}f = T^{\langle\alpha+\beta\rangle}f$.

THEOREM 5. *If $\alpha > 0$ and $f \in X[0, 1)$, the following statements are equivalent*:

(1) $T^{\langle\alpha\rangle}f = g \in X[0, 1)$.

(2) *There exists $g \in X[0, 1)$ such that $g\hat{}(k) = k^\alpha f\hat{}(k)$, $k \in \mathbb{P}$.*

(3) *There exists $g \in X[0, 1)$ such that $f = T^{\langle-\alpha\rangle}g + f\hat{}(0)$*

(*cf. Corollary* 1 *and* 3 *in* [4]).

Proof. Assertion (2) follows from (1) by Theorem 1, (3) follows from (2) by Lemma 6, and (1) follows from (3) by Theorem 4.

COROLLARY 5.1. $T^{\langle 1\rangle}f \in X[0, 1) \Leftrightarrow D^{[1]}f \in X[0, 1)$; *in this event* $T^{\langle 1\rangle}f = D^{[1]}f$.

Proof. By Theorem 5 $T^{\langle 1\rangle}f = g \in X[0, 1)$ is equivalent to $g\hat{}(k) = k \cdot f\hat{}(k)$ $(k \in \mathbb{P})$. On the other hand it is proved in [6] that the last equality is equivalent to $D^{[1]}f = g \in X[0, 1)$.

3. Applications

First we give a generalization of the Jackson-type and Bernstein-type theorems given in the case $\alpha = 0$ by Watari [10], and $\alpha \in \mathbb{N}$ by Butzer and Wagner [2, 8].

THEOREM 6. *If* $T^{\langle\alpha\rangle}f \in \mathrm{Lip}(X, \beta)$, $\alpha \geqslant 0$, $\beta > 0$, *then* $E_n(X, f) = O(1/n^{\alpha+\beta})$.

Proof. Let $M_r \leqslant n < M_{r+1}$; then by [14] and Lemma 5 we have

$$E_n(f) \leqslant E_{M_r}(f) \leqslant \omega\left(f, \frac{1}{M_r}\right) = O(1)\frac{1}{M_r^\alpha}\,\omega\left(T^{\langle\alpha\rangle}f, \frac{1}{M_r}\right)$$
$$= O\left(\frac{1}{M_r^{\alpha+\beta}}\right) = O\left(\frac{1}{n^{\alpha+\beta}}\right).$$

THEOREM 7. *If* $f \in X[0, 1)$, $E_n(X, f) = O(1/n^\beta)$, $\beta > \alpha \geqslant 0$, *then* $T^{\langle\alpha\rangle}f \in \mathrm{Lip}(X, \beta - \alpha)$.

Proof. Let $M_r \leqslant n < M_{r+1}$; then $\omega(f, 1/M_r) \leqslant 2E_{M_r}(f) = O(1/M_r^\beta)$. By Corollary 3.1, we have $T^{\langle\alpha\rangle}f \in X[0, 1)$. Moreover, by Lemma 5,

$$\omega\left(T^{\langle\alpha\rangle}, \frac{1}{M_r}\right) = O(1)\sum_{l=r}^{\alpha} \omega\left(f, \frac{1}{M_l}\right) M_l^\alpha = O(1)\sum_{l=r}^{\alpha} E_{M_l}(f)\, M_l^\alpha$$
$$= O(1)\sum_{l=r}^{\alpha} M_l^{\alpha-\beta} = O\left(\frac{1}{M_r^{\beta-\alpha}}\right).$$

For any $\delta > 0$, let $1/M_{r+1} \leqslant \delta < 1/M_r$; then $\omega(T^{\langle\alpha\rangle}f, \delta) \leqslant \omega(T^{\langle\alpha\rangle}f, 1/M_r) = O(1/M_r^{\beta-\alpha}) = O(\delta^{\beta-\alpha})$, i.e., $T^{\langle\alpha\rangle}f \in \mathrm{Lip}(X, \beta - \alpha)$.

Below we discuss the degree of approximation by the typical means of the Walsh–Fourier series.

DEFINITION 2. Let $f \in X[0, 1)$, $K_{n,\lambda}(t) = \sum_{k=0}^{n-1} [1 - (k/n)^\lambda]\, \overline{\psi_k(t)}$, $\lambda > 0$, then

$$R_{n,\lambda}(f, x) = f * K_{n,\lambda}(x) = \int_0^1 f(x \oplus t)\, K_{n,\lambda}(t)\, dt$$

are called the typical means of the Walsh–Fourier series of f.

THEOREM 8. *If* $T^{\langle\alpha\rangle}f \in X[0, 1)$, $\alpha \geqslant 0$, $\lambda > 0$, *then, for* $M_s \leqslant n < M_{s+1}$, $\|R_{n,\lambda}(f, \cdot) - f(\cdot)\|_X = O(1)(1/n^\lambda) \sum_{l=0}^{s} \omega(T^{\langle\alpha\rangle}f, 1/M_l)\, M_l^{\lambda-\alpha}$.

Proof. Since

$$\begin{aligned}
&\|R_{n,\lambda}(f,\cdot)-f(\cdot)\|_X \\
&\leqslant \left\|f*\sum_{k=0}^{M_s-1}\bar\psi_k-f\right\|_X+\left\|f*\sum_{k=M_s}^{a_sM_s-1}\bar\psi_k\right\|_X+\frac{1}{n^\lambda}\left\|f*\sum_{k=0}^{M_s-1}k^\lambda\bar\psi_k\right\|_X \\
&\quad+\frac{1}{n^\lambda}\left\|f*\sum_{k=M_s}^{a_sM_s-1}k^\lambda\bar\psi_k\right\|_X+\frac{1}{n^\lambda}\left\|f*\sum_{k=a_sM_s}^{n-1}(n^\lambda-k^\lambda)\,\bar\psi_k\right\|_X \\
&\leqslant \omega\left(\frac{1}{M_s}\right)+\left\|f*\sum_{k=M_s}^{a_sM_s-1}\bar\psi_k\right\|_X+\frac{O(1)}{n^\lambda}\sum_{l=0}^{s-1}\omega\left(f,\frac{1}{M_l}\right)M_l^\lambda \\
&\quad+\frac{1}{n^\lambda}\left\|f*\sum_{k=M_s}^{a_sM_s-1}k^\lambda\bar\psi_k\right\|_X+O(1)\,\omega\left(\frac{1}{M_s}\right),
\end{aligned}$$

$$\begin{aligned}
&\left\|f*\sum_{k=M_s}^{a_sM_s-1}\bar\psi_k\right\|_X \\
&\quad=\left\|f*\left(\sum_{j=1}^{a_s-1}\bar\varphi_s^j\right)D_{M_s}\right\|_X\leqslant(a_s-1)\,\omega\left(\frac{1}{M_s}\right).
\end{aligned}$$

by (7), as well as

$$\begin{aligned}
\left\|\sum_{k=M_s}^{a_sM_s-1}k^\lambda\bar\psi_k\right\|_X&=\left\|f*\sum_{j=1}^{a_s-1}\sum_{k=jM_s}^{(j+1)M_s-1}k^\lambda\bar\psi_k\right\|_X \\
&=\left\|\sum_{j=1}^{a_s-1}f*\bar\varphi_s^j\sum_{k=0}^{M_s-1}(jM_s+k)^\lambda\psi_k\right\|_X \\
&=O(1)\omega(1/M_s)\,M_s^\lambda,
\end{aligned}$$

it follows that $\|R_{n,\lambda}(f,\cdot)-f(\cdot)\|_X=(O(1)/n^\lambda)\sum_{l=0}^{s}\omega(f,1/M_l)\,M_l^\lambda=(O(1)/n^\lambda)\sum_{l=0}^{s}\omega(T^{\langle\alpha\rangle}f,1/M_l)\,M_l^{\lambda-\alpha}$.

THEOREM 9. *If* $f\in X[0,1)$, $\lambda>0$, *then* $\lambda_n=\|R_{n,\lambda}(f,\cdot)-f(\cdot)\|_X\to 0$ $(n\to\infty)$.

Proof. Let $M_s\leqslant n<M_{s+1}$. Since $(1/n^\lambda)\sum_{l=0}^{s}M_l^\lambda\leqslant\sum_{l=0}^{s}(M_l/M_s)^\lambda\leqslant 1+(1/2^\lambda)+(1/4^\lambda)+\cdots+(1/2^{s\lambda})\leqslant c<\infty$, we have $\Delta_n=O(1)(1/n^\lambda)\sum_{l=0}^{s}\omega(f,1/M_l)\,M_l^\lambda=O(1)(1/\sum_{l=0}^{s}M_l^\lambda)\sum_{l=0}^{s}\omega(f,1/M_l)\,M_l^\lambda$. It is easy to see that the transformation of the sequence $\{\omega(1/M_l)\}$ into the sequence $\{(1/\sum_{l=0}^{s}M_l^\lambda)\sum_{l=0}^{s}\omega(f,1/M_l)\,M_l^\lambda\}$ is regular. Therefore, since $\omega(1/M_l)\to 0$ $(l\to\infty)$, we get $\Delta_n\to 0$ $(n\to\infty)$.

THEOREM 10. *If* $T^{\langle\alpha\rangle}f \in \mathrm{Lip}(X, \beta)$, $\alpha \geqslant 0$, $\beta > 0$, $\lambda > 0$, *then*

$$\begin{aligned} \|R_{n,\lambda}(f,\cdot) - f(\cdot)\|_X &= O\left(\frac{1}{n^{\alpha+\beta}}\right) && \text{if } \alpha+\beta<\lambda, \\ &= O\left(\frac{\ln n}{n^{\lambda}}\right) && \text{if } \alpha+\beta=\lambda, \\ &= O\left(\frac{1}{n^{\lambda}}\right) && \text{if } \alpha+\beta>\lambda. \end{aligned}$$

Proof. Let $M_s \leqslant n < M_{s+1}$; then by Theorem 8, $\Delta_n = \|R_{n,\lambda}(f,\cdot) - f(\cdot)\|_X = (\mathrm{O}(1)/n^{\lambda}) \sum_{l=0}^{s} \omega(T^{\langle\alpha\rangle}f, 1/M_l)\, M_l^{\lambda-\alpha}$. If $\alpha+\beta<\lambda$, then

$$\begin{aligned} \Delta_n &= \frac{O(1)}{n^{\lambda}} \sum_{l=0}^{s} \frac{1}{M_l^{\beta}} M_l^{\lambda-\alpha} = \frac{O(1)}{n^{\lambda}} \sum_{l=0}^{s} M_l^{\lambda-\alpha-\beta} \\ &= \frac{O(1)}{M_s^{\alpha+\beta}} \sum_{l=0}^{s} \left(\frac{M_l}{M_s}\right)^{\lambda-\alpha-\beta} = O\left(\frac{1}{n^{\alpha+\beta}}\right). \end{aligned}$$

If $\alpha+\beta=\lambda$, then

$$\Delta_n = \frac{O(1)}{n^{\lambda}} \sum_{l=0}^{s} M_l^{0} = O\left(\frac{s}{n^{\lambda}}\right) = O\left(\frac{\ln n}{n^{\lambda}}\right).$$

If $\alpha+\beta>\lambda$, then

$$\Delta_n = \frac{O(1)}{n^{\lambda}} \sum_{l=0}^{s} \frac{1}{M_l^{\beta}} M_l^{\lambda-\alpha} = \frac{O(1)}{n^{\lambda}} \sum_{l=0}^{s} M_l^{\lambda-\alpha-\beta} = O\left(\frac{1}{n^{\lambda}}\right).$$

The above results have been obtained in the case $\alpha=0$, $0<\beta<1$, $\lambda=1$ by Yano [9] and Ефимов [14], and for $\alpha=0$, $\beta=1$, $\lambda=1$ by блюмин [12].

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